A GENERAL CLASS OF DISTRIBUTIONFREE TESTS FOR SYMMETRY CONTAINING THE TESTS OF WILCOXON AND FISHER *). II

BY

CONSTANCE VAN EEDEN AND A. BENARD

(Communicated by Prof. D. VAN DANTZIG at the meeting of March 30, 1957)

4. *The relation with the class of two sample tests described in* [4]

From (2.3) it follows that $\boldsymbol{T}$ may be written in the form

$$\boldsymbol{T}=2\sum_{i=1}^{k}\varphi_i\,\boldsymbol{a}_i-\sum_{i=1}^{k}\varphi_i\,t_i=2\mathbf{t}^*-\sum_{i=1}^{k}\varphi_i\,t_i, \tag{4.1}$$

where $\mathbf{t}^*$ is the test statistic for the two sample problem defined in [4] (p. 251) applied to the positive observations as the first sample and the absolute values of the negative observations as the second sample.

Further if (cf. e.g. [4] p. 252)

$$\tilde{\mathbf{t}}^*\overset{\text{def}}{=}\mathbf{t}^*-\mathscr{E}\,(\mathbf{t}^*\,|\,(k,t,u),\,\boldsymbol{n}_1;\,H_0)=\mathbf{t}^*-\frac{\boldsymbol{n}_1}{n}\sum_{i=1}^{k}\varphi_i\,t_i, \tag{4.2}$$

then

$$\boldsymbol{T}=2\tilde{\mathbf{t}}^*+\frac{2}{n}\,(\boldsymbol{n}_1-\tfrac{1}{2}n)\sum_{i=1}^{k}\varphi_i\,t_i. \tag{4.3}$$

Thus the test statistic $\boldsymbol{T}$ is a combination of the statistic $\mathbf{t}^*$ for the two sample problem and the statistic $\boldsymbol{n}_1$ of the sign test.

Special cases

For WILCOXON's test for symmetry we obtain from (4.3)

$$\boldsymbol{T}_W=\tilde{\boldsymbol{W}}+(n+1)\,(\boldsymbol{n}_1-\tfrac{1}{2}n), \tag{4.4}$$

with

$$\tilde{\boldsymbol{W}}\overset{\text{def}}{=}\boldsymbol{W}-\boldsymbol{n}_1\,\boldsymbol{n}_2, \tag{4.5}$$

where $\boldsymbol{W}$ is the test statistic of WILCOXON's two sample test [7]).

In the case of FISHER's test for symmetry we have

$$\boldsymbol{T}_F=2\tilde{\mathbf{t}}_P+\frac{2}{n}\,(\boldsymbol{n}_1-\tfrac{1}{2}n)\sum_{h=1}^{m}|z_h|, \tag{4.6}$$

where $\mathbf{t}_P$ is E. J. G. PITMAN's test statistic for the two sample problem [13].

*) Report SP 54 of the Statistical Department of the Mathematical Centre, Amsterdam.

[7]) The test statistic of WILCOXON's two sample test for the samples $x_1, \ldots, x_{n_1}$ and $y_1, \ldots, y_{n_2}$ is defined here as twice the number of pairs (x_i, y_j) with $x_i > y_j$, increased by the number of pairs (x_i, y_j) with $x_i = y_j$ $(i = 1, \ldots, n_1;\ j = 1, \ldots, n_2)$ (cf. [20]).

Remark

6. Other tests for symmetry may e.g. be obtained by choosing for $\mathbf{t}^*$ the test statistic of the two sample tests of M. E. TERRY [18] or B. L. VAN DER WAERDEN [21], i.e. by taking

$$\varphi_i = \frac{1}{t_i} \sum_{\gamma=1}^{t_i} \mathscr{E} \mathbf{Z}_{n, s_i+\gamma} \quad (i=1, \ldots, k) \tag{4.7}$$

or

$$\varphi_i = \frac{1}{t_i} \sum_{\gamma=1}^{t_i} \Psi\left(\frac{s_i+\gamma}{n+1}\right) \quad (i=1, \ldots, k), \tag{4.8}$$

with

$$s_i \overset{\text{def}}{=} \sum_{j=1}^{i-1} t_j \quad (i=1, \ldots, k) \tag{4.9}$$

and where $\mathscr{E} \mathbf{Z}_{n,r}$ is the expectation of the r-th order statistic of a random sample of size n from a standard normal distribution and $\Psi(x)$ is defined by

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Psi(x)} e^{-\frac{1}{2}u^2} du = x. \tag{4.10}$$

Further the hypothesis H_0 implies, under the conditions (k, t, u) and $\mathbf{n}_1 = n_1$ the hypothesis H_0'' that the positive observations are a random sample without replacement taken from the absolute values of all observations (cf. [9] p. 71 and [5] p. 307). The mean and variance of $\mathbf{T}$ under the hypothesis H_0 and under the condition (k, t, u) thus also follow from the formulae for the mean and variance of $\mathbf{t}^*$ under the hypothesis H_0'' (cf. e.g. [4] p. 252).

From (4.3) it follows

$$\mathscr{E}(\mathbf{T} \mid (k, t, u), n_1; H_0) = \frac{2}{n} (n_1 - \tfrac{1}{2} n) \sum_{i=1}^{k} \varphi_i t_i \tag{4.11}$$

and

$$\sigma^2(\mathbf{T} \mid (k, t, u), n_1; H_0) = \frac{4 n_1 n_2}{n(n-1)} \left\{ \sum_{i=1}^{k} t_i \varphi_i^2 - \frac{1}{n} (\sum_{i=1}^{k} t_i \varphi_i)^2 \right\}. \tag{4.12}$$

From (4.11) and (4.12) then follows

$$\left\{ \begin{aligned} &\mathscr{E}(\mathbf{T} \mid (k, t, u); H_0) = \mathscr{E}\{\mathscr{E}(\mathbf{T} \mid (k, t, u), \mathbf{n}_1; H_0) \mid (k, t, u); H_0\} = \\ &= \frac{2}{n} \sum_{i=1}^{k} t_i \varphi_i \mathscr{E}(\mathbf{n}_1 - \tfrac{1}{2} n \mid (k, t, u); H_0) = 0 \qquad \text{(cf. (3.24))} \end{aligned} \right. \tag{4.13}$$

and

$$\left\{ \begin{aligned} &\sigma^2(\mathbf{T} \mid (k, t, u); H_0) = \\ &= \sigma^2\{\mathscr{E}(\mathbf{T} \mid (k, t, u), \mathbf{n}_1; H_0) \mid (k, t, u); H_0\} + \\ &+ \mathscr{E}\{\sigma^2(\mathbf{T} \mid (k, t, u), \mathbf{n}_1; H_0) \mid (k, t, u); H_0\} = \\ &= \frac{4}{n^2} \{\sum_{i=1}^{k} t_i \varphi_i\}^2 \sigma^2(\mathbf{n}_1 \mid (k, t, u); H_0) + \\ &+ \frac{4}{n(n-1)} \{\sum_{i=1}^{k} t_i \varphi_i^2 - \frac{1}{n} (\sum_{i=1}^{k} t_i \varphi_i)^2\} \mathscr{E}(\mathbf{n}_1 \mathbf{n}_2 \mid (k, t, u); H_0) = \\ &= \frac{1}{n} \{\sum_{i=1}^{k} t_i \varphi_i\}^2 + \{\sum_{i=1}^{k} t_i \varphi_i^2 - \frac{1}{n} (\sum_{i=1}^{k} t_i \varphi_i)^2\} = \sum_{i=1}^{k} t_i \varphi_i^2 \quad \text{(cf. (3.25))}. \end{aligned} \right. \tag{4.14}$$

5. *The consistency of the tests of* WILCOXON *and* FISHER

In this section the consistency of the tests for symmetry of WILCOXON and FISHER will be investigated.

We again consider the sequence $\{\mathbf{z}_\lambda\}$ and an alternative hypothesis H stating that the distributions of $\mathbf{z}_\lambda$ under the condition $\mathbf{z}_\lambda \neq 0$ are, for $\lambda = 1, 2, \ldots,$ identical. Let $\mathbf{x}_{1,\lambda}, \ldots, \mathbf{x}_{\mathbf{n}_{1,\lambda},\lambda}$ denote the positive observations and $\mathbf{y}_{1,\lambda}, \ldots, \mathbf{y}_{\mathbf{n}_{2,\lambda},\lambda}$ the absolute values of the negative observations, with $\mathbf{n}_{1,\lambda} + \mathbf{n}_{2,\lambda} = \mathbf{n}_\lambda$. Let further

$$
(5.1) \qquad \begin{cases} p \overset{\text{def}}{=} \mathrm{P}\,[\mathbf{z}_\lambda > 0 \mid \mathbf{z}_\lambda \neq 0] \quad (\lambda = 1, 2, \ldots), \\ q \overset{\text{def}}{=} 1 - p. \end{cases}
$$

We first prove the consistency of WILCOXON's test. Let

$$
(5.2) \qquad \theta \overset{\text{def}}{=} \mathrm{P}\,[\mathbf{x}_\lambda > \mathbf{y}_\mu] - \mathrm{P}\,[\mathbf{x}_\lambda < \mathbf{y}_\mu] \quad (\lambda, \mu = 1, 2, \ldots),
$$

then we have

Lemma I:

$$
(5.3) \qquad \mu_W \overset{\text{def}}{=} \mathscr{E}\,(\mathbf{T}_W \mid n; H) = n\,(n-1)\,pq\theta + n\,(n+1)\,(p - \tfrac{1}{2}).
$$

Proof: From (4.4) it follows that

$$
(5.4) \qquad \mathbf{T}_W = \sum_{i=1}^{\mathbf{n}_1} \sum_{j=1}^{\mathbf{n}_2} sgn\,(\mathbf{x}_i - \mathbf{y}_j) + (n+1)\,(\mathbf{n}_1 - \tfrac{1}{2} n),
$$

where

$$
(5.5) \qquad sgn\, z = \begin{cases} 1 \text{ if } z > 0 \\ 0 \text{ if } z = 0 \\ -1 \text{ if } z < 0. \end{cases}
$$

From (5.4) follows

$$
(5.6) \qquad \mathscr{E}(\mathbf{T}_W | n, n_1; H) = n_1\, n_2 \theta + (n+1)(n_1 - \tfrac{1}{2} n),
$$

thus

$$
(5.7) \qquad \begin{cases} \mathscr{E}(\mathbf{T}_W | n; H) = \mathscr{E}\{\mathscr{E}(\mathbf{T}_W | n, \mathbf{n}_1; H) | n; H\} = \\ = \theta \mathscr{E}(\mathbf{n}_1 \mathbf{n}_2 | n; H) + (n+1)\mathscr{E}(\mathbf{n}_1 - \tfrac{1}{2} n | n; H) = \\ = n(n-1) pq\theta + n(n+1)(p - \tfrac{1}{2}). \end{cases}
$$

Lemma II:

$$
(5.8) \qquad \sigma_W^2 \overset{\text{def}}{=} \sigma^2\,(\mathbf{T}_W \mid n; H) = \mathrm{O}\,(n^3)
$$

and the coefficient of n^3 *in* (5.8) *is* $\leq \frac{13}{16}$.

Proof: We have

$$
(5.9) \quad \sigma^2(\mathbf{T}_W | n; H) = \sigma^2\{\mathscr{E}(\mathbf{T}_W | n, \mathbf{n}_1; H) | n; H\} + \mathscr{E}\{\sigma^2(\mathbf{T}_W | n, \mathbf{n}_1; H) | n; H\}.
$$

From (5.6) it follows that

$$
(5.10) \quad \sigma^2\{\mathscr{E}(\mathbf{T}_W | n, \mathbf{n}_1; H) | n; H\} = \sigma^2\{\mathbf{n}_1\, \mathbf{n}_2 \theta + (n+1)(\mathbf{n}_1 - \tfrac{1}{2} n) | n; H\} = \mathrm{O}(n^3)
$$

and the coefficient of n^3 in (5.10) is

$$
(5.11) \qquad pq(\theta + 1 - 2pq\theta)^2.
$$

Further (cf. (5.4))

$$\sigma^2(\mathbf{T}_W \mid n, n_1; H) = \sigma^2(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} sgn\,(\mathbf{x}_i - \mathbf{y}_j) \mid n, n_1; H) \tag{5.12}$$

and from D. J. STOKER ([17], p. 67–68) it follows that

$$\sigma^2(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} sgn\,(\mathbf{x}_i - \mathbf{y}_j) \mid n, n_1; H) \leqq n_1 n_2 (n+1), \tag{5.13}$$

thus

$$\mathscr{E}\{\sigma^2(\mathbf{T}_W|n, \mathbf{n}_1; H)|n; H\} \leqq n(n^2-1)pq. \tag{5.14}$$

Thus

$$\sigma^2(\mathbf{T}_W|n; H) = \mathrm{O}(n^3) \tag{5.15}$$

and the coefficient of n^3 in (5.15) is

$$\leqq pq(\theta+1-2pq\theta)^2 + pq \leqq \tfrac{13}{16}\,{}^{8)}. \tag{5.16}$$

Theorem VI: *If (3.47) is satisfied then the test for symmetry of* WILCOXON *based on the critical region Z (cf. (2.5)) is, for $\lambda \to \infty$, consistent for the class of alternative hypotheses*

$$|p-\tfrac{1}{2}+pq\theta| > 0. \tag{5.17}$$

The tests based on the critical regions Z_l and Z_r respectively are consistent for the classes of alternative hypotheses

$$p-\tfrac{1}{2}+pq\theta < 0 \tag{5.18}$$

and

$$p-\tfrac{1}{2}+pq\theta > 0 \tag{5.19}$$

respectively and not consistent for the classes of alternative hypotheses

$$p-\tfrac{1}{2}+pq\theta > 0 \tag{5.20}$$

and

$$p-\tfrac{1}{2}+pq\theta < 0 \tag{5.21}$$

respectively.

All tests of WILCOXON *mentioned are, for sufficiently small α, not consistent for the class of alternative hypotheses*

$$p-\tfrac{1}{2}+pq\theta = 0. \tag{5.22}$$

Proof: [9]) The index λ will be omitted. Let

$$\begin{cases} 1. & \mathbf{s}_W^2 \overset{\text{def}}{=} \sigma^2(\mathbf{T}_W \mid n, \mathbf{t}_1, \ldots, \mathbf{t}_k; H_0), \\ 2. & c_1^2 \overset{\text{def}}{=} \tfrac{1}{4} n (n+1)^2 \\ 3. & c_2^2 \overset{\text{def}}{=} \tfrac{1}{6} n (n+1)(2n+1), \end{cases} \tag{5.23}$$

---

[8]) If $p = \frac{1}{2}$ and $\theta = 1$ then

$$pq(\theta+1-2pq\theta)^2 = \tfrac{9}{16}$$

and (cf. [17]. p. 67–68)

$$\sigma^2(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} sgn\,(\mathbf{x}_i - \mathbf{y}_j) \mid n, n_1; H) = n_1 n_2 (n-2).$$

Thus in this case the coefficient of n^3 in (5.15) equals $\frac{13}{16}$.

[9]) Cf. also D. VAN DANTZIG [3] for the proof of the consistency of WILCOXON's two sample test.

then

$$c_1^2 \leqq \mathbf{s}_W^2 \leqq c_2^2. \tag{5.24}$$

We first consider the case that

$$p-\tfrac{1}{2}+pq\theta<0. \tag{5.25}$$

For the test based on Z_l we have (cf. lemma I and II)

$$\begin{cases} \lim\limits_{\lambda\to\infty} \mathrm{P}\,[\boldsymbol{T}_W \notin \boldsymbol{Z}_l \,|\, n; H] = \lim\limits_{\lambda\to\infty} \mathrm{P}\,[\boldsymbol{T}_W > -\xi_\alpha \mathbf{s}_W \,|\, n; H] \leqq \\ \leqq \lim\limits_{\lambda\to\infty} \mathrm{P}\left[\frac{\boldsymbol{T}_W-\mu_W}{\sigma_W} > -\frac{\xi_\alpha c_2+\mu_W}{\sigma_W} \,\Big|\, n; H\right], \end{cases} \tag{5.26}$$

where ξ_α is defined by

$$\frac{1}{\sqrt{2\pi}} \int\limits_{\xi_\alpha}^{\infty} e^{-\frac{1}{2}x^2}\, dx = \alpha. \tag{5.27}$$

From (5.23), (5.25), lemma II and the fact that n tends to infinity with λ it follows that $-\frac{\xi_\alpha c_2+\mu_W}{\sigma_W}$ is positive for sufficiently large λ; thus according to the inequality of BIENAYMÉ–TCHEBYCHEF

$$\lim_{\lambda\to\infty} \mathrm{P}\,[\boldsymbol{T}_W \notin \boldsymbol{Z}_l \,|\, n; H] \leq \lim_{\lambda\to\infty} \frac{\sigma_W^2}{(\xi_\alpha c_2+\mu_W)^2} = 0. \tag{5.28}$$

Thus the test based on the critical region Z_l is, for $\lambda \to \infty$, consistent for the class of alternative hypotheses (5.25).

If

$$p-\tfrac{1}{2}+pq\theta>0 \tag{5.29}$$

then

$$\begin{cases} \lim\limits_{\lambda\to\infty} \mathrm{P}\,[\boldsymbol{T}_W \in \boldsymbol{Z}_l \,|\, n; H] \leq \lim\limits_{\lambda\to\infty} \mathrm{P}\,[\boldsymbol{T}_W \leqq -\xi_\alpha c_1 \,|\, n; H] \leqq \\ \leq \lim\limits_{\lambda\to\infty} \frac{\sigma_W^2}{(\xi_\alpha c_1+\mu_W)^2} = 0, \end{cases} \tag{5.30}$$

$-\frac{\xi_\alpha c_1+\mu_W}{\sigma_W}$ being negative for sufficiently large λ. Thus the test based on Z_l is, for $\lambda \to \infty$, not consistent for the class of alternative hypotheses (5.29).

Finally if

$$p-\tfrac{1}{2}+pq\theta=0 \tag{5.31}$$

then

$$\begin{cases} \lim\limits_{\lambda\to\infty} \mathrm{P}\,[\boldsymbol{T}_W \in \boldsymbol{Z}_l \,|\, n, H] \leqq \lim\limits_{\lambda\to\infty} \mathrm{P}\left[\frac{\boldsymbol{T}_W-\mu_W}{\sigma_W} \leqq -\frac{\xi_\alpha c_1+\mu_W}{\sigma_W} \,\Big|\, n; H\right] \leqq \\ \leqq \lim\limits_{\lambda\to\infty} \left(\frac{\sigma_W}{\xi_\alpha c_1}\right)^2. \end{cases} \tag{5.32}$$

Thus if

$$\xi_\alpha > \lim_{\lambda\to\infty} \frac{\sigma_W}{c_1} \tag{5.33}$$

then the test based on Z_l is, for $\lambda \to \infty$, not consistent for the class of alternative hypotheses (5.31) and from (5.23) and lemma II follows

$$\lim_{\lambda\to\infty} \frac{\sigma_W}{c_1} \leqq \sqrt{3{,}25} = 1{,}80. \tag{5.34}$$

The proofs for the tests based on Z_r and Z are analogous.

Theorem VII: *If the distributions of* $\mathbf{z}_1, \ldots, \mathbf{z}_m$ *are identical and symmetrical with respect to* a *then*

$$\begin{cases} 1.\ p-\frac{1}{2}+pq\theta=0 \text{ if } a=0, \\ 2.\ (p-\frac{1}{2}+pq\theta)a>0 \text{ if } a\neq 0. \end{cases} \tag{5.35}$$

Proof: Let

$$H(z) \overset{\text{def}}{=} \mathrm{P}\,[\mathbf{z}_h \leqq z] \tag{5.36}$$

and (cf. (3.46))

$$\pi \overset{\text{def}}{=} \mathrm{P}\,[\mathbf{z}_h \neq 0]. \tag{5.37}$$

Then (cf. (5.1))

$$p=\frac{1}{\pi}\int_0^\infty dH(z), \qquad q=\frac{1}{\pi}\int_{-\infty}^{0-} dH(z). \text{ [10]} \tag{5.38}$$

If $a=0$ then $p=\frac{1}{2}$ and $\theta=0$, thus

$$p-\tfrac{1}{2}+pq\theta=0 \text{ if } a=0. \tag{5.39}$$

Now consider the case that $a>0$; then $p \geqq \frac{1}{2}$. From the fact that the distribution of $\mathbf{z}_h$ is symmetrical with respect to a it follows that

$$q=\frac{1}{\pi}\int_{2a}^\infty dH(z). \tag{5.40}$$

If further

$$F(x) \overset{\text{def}}{=} \mathrm{P}\,[\mathbf{x}_i \leqq x], \qquad G(y) \overset{\text{def}}{=} \mathrm{P}\,[\mathbf{y}_j \leqq y] \tag{5.41}$$

then

$$dF(x)=\frac{dH(x)}{p}, \qquad F(x)=\frac{1}{p}\int_0^x dH(u) \tag{5.42}$$

and from the symmetry of the distribution of $\mathbf{z}_h$ with respect to a it follows that

$$dG(y)=\frac{dH(y+2a)}{q}, \qquad G(y)=\frac{1}{q}\int_{2a}^{2a+y} dH(u). \tag{5.43}$$

---

[10]) Here we define

$$\int_{z_1}^{z_2} dH(z) \overset{\text{def}}{=} \mathrm{P}\,[z_1 < \mathbf{z} \leqq z_2]$$

and

$$\int_{z_1}^{z_2-} dH(z) \overset{\text{def}}{=} \mathrm{P}\,[z_1 < \mathbf{z} < z_2].$$

If $q > 0$ then

$$(5.44)\quad \begin{cases} \theta = \mathrm{P}[\mathbf{x}_i > \mathbf{y}_j] - \mathrm{P}[\mathbf{x}_i < \mathbf{y}_j] > \mathrm{P}[\mathbf{x}_i > \mathbf{y}_j + 2a] - \mathrm{P}[\mathbf{x}_i < \mathbf{y}_j + 2a] = \\ = \frac{1}{pq}\{\int\limits_{2a}^{\infty} dH(x) \int\limits_{2a}^{x} dH(u) - \int\limits_{0}^{\infty} dH(x+2a) \int\limits_{0}^{x+2a} dH(u)\} \end{cases}$$

and from (5.44) follows

$$(5.45)\quad \begin{cases} pq\theta > \int\limits_{2a}^{\infty} dH(x) \int\limits_{2a}^{x} dH(u) - \int\limits_{0}^{\infty} dH(x+2a) \int\limits_{0}^{x+2a} dH(u) = \\ = \int\limits_{2a}^{\infty} dH(x) \int\limits_{2a}^{x} dH(u) - \int\limits_{2a}^{\infty} dH(x) \int\limits_{0}^{x} dH(u) = \\ = \int\limits_{2a}^{\infty} dH(x) \int\limits_{2a}^{\infty} dH(u) - \int\limits_{2a}^{\infty} dH(x) \int\limits_{0}^{\infty} dH(u) = \\ = \pi^2 q^2 - \pi^2 pq = \pi^2 q(q-p). \end{cases}$$

Thus if $q > 0$ then

$$(5.46)\quad \begin{cases} p - \frac{1}{2} + pq\theta > p - \frac{1}{2} + \pi^2 q(q-p) = (p-q)(\frac{1}{2} - \pi^2 q) \geqq \\ \geqq (p-q)(\frac{1}{2} - q) = \frac{1}{2}(p-q)^2 \geqq 0. \end{cases}$$

Further if $q = 0$ then $p = 1$ and

$$(5.47)\qquad p - \tfrac{1}{2} + pq\theta = p - \tfrac{1}{2} > 0.$$

Thus $p - \frac{1}{2} + pq\theta$ is positive if a is positive.

The proof for $a < 0$ is analogous.

From the theorems VI and VII it follows that if the distributions of $\mathbf{z}_\lambda$ are, for $\lambda = 1, 2, \ldots$, identical and symmetrical with respect to a then WILCOXON's test for symmetry based on the critical region Z is, for $\lambda \to \infty$, consistent for the class of alternative hypotheses

$$(5.48)\qquad a \neq 0.$$

The tests based on Z_l and Z_r respectively are consistent for the classes of alternative hypotheses

$$(5.49)\qquad a < 0$$

and

$$(5.50)\qquad a > 0$$

respectively and not consistent for the classes of alternative hypotheses

$$(5.51)\qquad a > 0$$

and

$$(5.52)\qquad a < 0$$

respectively.

We now consider FISHER's test for symmetry.

Theorem VIII: *If* (3.47) *is satisfied and if*

$$(5.53)\qquad \mathscr{E}(\mathbf{z}_\lambda^2 | \mathbf{z}_\lambda \neq 0) < \infty$$

then FISHER's *test for symmetry based on the critical region Z is, for $\lambda \to \infty$, consistent for the class of alternative hypotheses*

$$\mathscr{E}(\mathbf{z}_\lambda | \mathbf{z}_\lambda \neq 0) \neq 0. \tag{5.54}$$

The tests based on the critical regions Z_l and Z_r respectively are consistent for the classes of alternative hypotheses

$$\mathscr{E}(\mathbf{z}_\lambda | \mathbf{z}_\lambda \neq 0) < 0 \tag{5.56}$$

and

$$\mathscr{E}(\mathbf{z}_\lambda | \mathbf{z}_\lambda \neq 0) > 0 \tag{5.57}$$

respectively and not consistent for the classes of alternative hypotheses

$$\mathscr{E}(\mathbf{z}_\lambda | \mathbf{z}_\lambda \neq 0) > 0 \tag{5.58}$$

and

$$\mathscr{E}(\mathbf{z}_\lambda | \mathbf{z}_\lambda \neq 0) < 0 \tag{5.59}$$

respectively.

All tests of FISHER *mentioned are, for sufficiently small α, not consistent for the class of alternative hypotheses*

$$\mathscr{E}(\mathbf{z}_\lambda | \mathbf{z}_\lambda \neq 0) = 0. \tag{5.60}$$

Proof: The index λ is omitted.

We have

$$\mu_F \stackrel{\text{def}}{=} \mathscr{E}(\mathbf{T}_F \mid n; H) = n \mathscr{E}(\mathbf{z} \mid \mathbf{z} \neq 0) \tag{5.61}$$

and

$$\sigma_F^2 \stackrel{\text{def}}{=} \sigma^2(\mathbf{T}_F \mid n; H) = n\, \sigma^2(\mathbf{z} \mid \mathbf{z} \neq 0). \tag{5.62}$$

We first consider the case that

$$\mathscr{E}(\mathbf{z} | \mathbf{z} \neq 0) < 0. \tag{5.63}$$

Let

$$\mathbf{s}_F^2 \stackrel{\text{def}}{=} \sigma^2(\mathbf{T}_F \mid n, \mathbf{t}_1, \ldots, \mathbf{t}_k, \mathbf{u}_1, \ldots, \mathbf{u}_k; H_0) = \sum_{h=1}^{m} \mathbf{z}_h^2, \tag{5.64}$$

then we have for each $\delta > 0$

$$\left\{\begin{aligned} &\lim_{\lambda\to\infty} \mathrm{P}\,[\mathbf{T}_F \notin \mathbf{Z}_l \mid n; H] = \lim_{\lambda\to\infty} \mathrm{P}\,[\mathbf{T}_F > -\xi_\alpha\, \mathbf{s}_F \mid n; H] = \\ &= \lim_{\lambda\to\infty} \mathrm{P}\,[\mathbf{T}_F > -\xi_\alpha\, \mathbf{s}_F \text{ and } |(1/n)\, \mathbf{s}_F^2 - \mathscr{E}(\mathbf{z}^2 \mid \mathbf{z} \neq 0)| < \delta \mid n; H] + \\ &+ \lim_{\lambda\to\infty} \mathrm{P}\,[\mathbf{T}_F > -\xi_\alpha\, \mathbf{s}_F \text{ and } |(1/n)\, \mathbf{s}_F^2 - \mathscr{E}(\mathbf{z}^2 \mid \mathbf{z} \neq 0)| \geqq \delta \mid n; H] \leqq \\ &\leqq \lim_{\lambda\to\infty} \mathrm{P}\,[\mathbf{T}_F > -\xi_\alpha \sqrt{n\{\mathscr{E}(\mathbf{z}^2 \mid \mathbf{z} \neq 0) + \delta\}} \mid n; H] + \\ &+ \lim_{\lambda\to\infty} \mathrm{P}\,[|(1/n)\, \mathbf{s}_F^2 - \mathscr{E}(\mathbf{z}^2 \mid \mathbf{z} \neq 0)| \geqq \delta \mid n; H]. \end{aligned}\right. \tag{5.65}$$

Further it follows from (5.53) (cf. also (3.65)) that the second term in the right hand member of (5.65) is zero; thus according to the inequality of BIENAYMÉ–TCHEBYCHEF we have

$$(5.66)\quad \begin{cases} \lim\limits_{\lambda\to\infty} \mathrm{P}\,[\mathbf{T}_F \notin \mathbf{Z}_l \mid n; H] \leqq \\ \leqq \lim\limits_{\lambda\to\infty} \mathrm{P}\left[\dfrac{\mathbf{T}_F-\mu_F}{\sigma_F} > -\dfrac{\xi_\alpha \sqrt{n\{\mathscr{E}(\mathbf{z}^2 \mid \mathbf{z}\neq 0)+\delta\}}+\mu_F}{\sigma_F} \mid n; H\right] \leqq \\ \leqq \lim\limits_{\lambda\to\infty} \dfrac{\sigma_F}{[\xi_\alpha \sqrt{n\{\mathscr{E}(\mathbf{z}^2 \mid \mathbf{z}\neq 0)+\delta\}}+\mu_F]^2} = 0, \end{cases}$$

$-\dfrac{\xi_\alpha \sqrt{n\{\mathscr{E}(\mathbf{z}^2 \mid \mathbf{z}\neq 0)+\delta\}}+\mu_F}{\sigma_F}$ being positive for sufficiently large λ. Thus the test based on Z_l is, for $\lambda \to \infty$, consistent for the class of alternative hypotheses (5.63).

In an analogous way it may be proved (cf. also the proof of theorem VI) that the test based on Z_l is not consistent for the class of alternative hypotheses

$$(5.67)\qquad \mathscr{E}(\mathbf{z} \mid \mathbf{z}\neq 0) > 0.$$

Finally if

$$(5.68)\qquad \mathscr{E}(\mathbf{z} \mid \mathbf{z}\neq 0) = 0$$

then we have (cf. (5.65) and (5.66)), for $0<\delta<\mathscr{E}(\mathbf{z}^2 \mid \mathbf{z}\neq 0)$,

$$(5.69)\quad \begin{cases} \lim\limits_{\lambda\to\infty} \mathrm{P}\,[\mathbf{T}_F \in \mathbf{Z}_l \mid n; H] \leqq \\ \leqq \lim\limits_{\lambda\to\infty} \mathrm{P}\left[\dfrac{\mathbf{T}_F-\mu_F}{\sigma_F} \leqq -\dfrac{\xi_\alpha \sqrt{n\{\mathscr{E}(\mathbf{z}^2 \mid \mathbf{z}\neq 0)-\delta\}}}{\sigma_F}\right] \leqq \\ \leqq \lim\limits_{\lambda\to\infty} \dfrac{\sigma_F^2}{\xi_\alpha^2 n\{\mathscr{E}(\mathbf{z}^2 \mid \mathbf{z}\neq 0)-\delta\}} = \dfrac{\mathscr{E}(\mathbf{z}^2 \mid \mathbf{z}\neq 0)}{\xi_\alpha^2\{\mathscr{E}(\mathbf{z}^2 \mid \mathbf{z}\neq 0)-\delta\}}. \end{cases}$$

Thus if

$$(5.70)\qquad \xi_\alpha > \sqrt{\frac{\mathscr{E}(\mathbf{z}^2 \mid \mathbf{z}\neq 0)}{\mathscr{E}(\mathbf{z}^2 \mid \mathbf{z}\neq 0)-\delta}}$$

then the test based on Z_l is not consistent for the class of alternative hypotheses (5.68).

The proofs for the tests based on Z_r and Z are analogous.

Remark

7. If

$$(5.71)\quad \begin{cases} \mu_1 \overset{\text{def}}{=} \mathscr{E}(\mathbf{x}_{i,\lambda}) = \mathscr{E}(\mathbf{z}_\lambda \mid \mathbf{z}_\lambda > 0), \\ \mu_2 \overset{\text{def}}{=} \mathscr{E}(\mathbf{y}_{j,\lambda}) = -\mathscr{E}(\mathbf{z}_\lambda \mid \mathbf{z}_\lambda < 0) \end{cases}$$

then (cf. (5.1))

$$(5.72)\qquad \mathscr{E}(\mathbf{z}_\lambda \mid \mathbf{z}_\lambda \neq 0) = p\mu_1 - q\mu_2.$$

Thus $\mathscr{E}(\mathbf{z}_\lambda \mid \mathbf{z}_\lambda\neq 0) \gtreqless 0$ is identical with

$$(5.73)\qquad p \gtreqless \frac{\mu_2}{\mu_1+\mu_2}.$$

(To be continued)